MATHEMATICS
Paper – II

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions:
There are EIGHT questions in all, out of which FIVE are to be attempted.
Questions no. 1 and 5 are compulsory. Out of the remaining SIX questions, THREE are to
be attempted selecting at least ONE question from each of the two Sections A and B.
Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a
question shall be counted even if attempted partly. Any page or portion of the page left
blank in the Question-cum-Answer Booklet must be clearly struck off.
All questions carry equal marks. The number of marks carried by a question/part is
indicated against it.
Answers must be written in ENGLISH only.
Unless otherwise mentioned, symbols and notations have their usual standard meanings.
Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) If in a group G there is an element \( a \) of order 360, what is the order of
\( a^{220} \)? Show that if G is a cyclic group of order \( n \) and \( m \) divides \( n \), then G
has a subgroup of order \( m \).

(b) Let \( \sum_{n=1}^{\infty} a_n \) be an absolutely convergent series of real numbers.

Suppose \( \sum_{n=1}^{\infty} a_{2n} = \frac{9}{8} \) and \( \sum_{n=0}^{\infty} a_{2n+1} = \frac{-3}{8} \). What is \( \sum_{n=1}^{\infty} a_n \)?

Justify your answer. (Majority of marks is for the correct justification).

(c) Let \( u(x, y) = \cos x \sinh y \). Find the harmonic conjugate \( v(x, y) \) of \( u \) and
express \( u(x, y) + i v(x, y) \) as a function of \( z = x + iy \).
(d) Solve graphically:
Maximize \( z = 7x + 4y \)
subject to \( 2x + y \leq 2, \ x + 10y \leq 10 \) and \( x \leq 8 \).
(Draw your own graph without graph paper).

Q2. 
(a) If \( p \) is a prime number and \( e \) a positive integer, what are the elements 'a' in the ring \( \mathbb{Z}_{p^e} \) of integers modulo \( p^e \) such that \( a^2 = a \)? Hence (or otherwise) determine the elements in \( \mathbb{Z}_{35} \) such that \( a^2 = a \).

(b) Let \( X = (a, b) \). Construct a continuous function \( f: X \rightarrow \mathbb{R} \) (set of real numbers) which is unbounded and not uniformly continuous on \( X \). Would your function be uniformly continuous on \( [a + \epsilon, b] \), \( a + \epsilon < b \)? Why?

(c) Evaluate the integral \( \int_{-1}^{1} \frac{z^2}{(z^2 + 1)(z - 1)^2} \, dz \), where \( r \) is the circle \( |z| = 2 \).

Q3. 
(a) What is the maximum possible order of a permutation in \( S_8 \), the group of permutations on the eight numbers \( \{1, 2, 3, \ldots, 8\} \)? Justify your answer. (Majority of marks will be given for the justification).

(b) Let \( f_n(x) = \frac{x}{1 + nx^2} \) for all real \( x \). Show that \( f_n \) converges uniformly to a function \( f \). What is \( f \)? Show that for \( x \neq 0 \), \( f_n(x) \rightarrow f'(x) \) but \( f'_n(0) \) does not converge to \( f'(0) \). Show that the maximum value \( |f_n(x)| \) can take is \( \frac{1}{2\sqrt{n}} \).

(c) A manufacturer wants to maximise his daily output of bulbs which are made by two processes \( P_1 \) and \( P_2 \). If \( x_1 \) is the output by process \( P_1 \) and \( x_2 \) is the output by process \( P_2 \), then the total labour hours is given by \( 2x_1 + 3x_2 \) and this cannot exceed 130, the total machine time is given by \( 3x_1 + 8x_2 \) which cannot exceed 300 and the total raw material is given by \( 4x_1 + 2x_2 \) and this cannot exceed 140. What should \( x_1 \) and \( x_2 \) be so that the total output \( x_1 + x_2 \) is maximum? Solve by the simplex method only.
Q4. (a) Compute the double integral which will give the area of the region between the y-axis, the circle \((x - 2)^2 + (y - 4)^2 = z^2\) and the parabola \(2y = x^2\). Compute the integral and find the area.

(b) Show that \(\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx = \frac{\pi}{\sqrt{2}}\) by using contour integration and the residue theorem.

(c) Solve the following transportation problem:

<table>
<thead>
<tr>
<th></th>
<th>D_1</th>
<th>D_2</th>
<th>D_3</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>O_1</td>
<td>5</td>
<td>3</td>
<td>6</td>
<td>20</td>
</tr>
<tr>
<td>O_2</td>
<td>4</td>
<td>7</td>
<td>9</td>
<td>40</td>
</tr>
<tr>
<td>Demand</td>
<td>15</td>
<td>22</td>
<td>23</td>
<td>60</td>
</tr>
</tbody>
</table>
SECTION B

Q5. (a) Store the value of \(-1\) in hexadecimal in a 32-bit computer.

(b) Show that \(\sum_{k=1}^{n} l_k(x) = 1\), where \(l_k(x), \ k = 1\) to \(n\), are Lagrange’s fundamental polynomials.

(c) Derive the Hamiltonian and equation of motion for a simple pendulum.

(d) Find the solution of the equation \(u_{xx} - 3u_{xy} + u_{yy} = \sin(x - 2y)\).

Q6. (a) Solve the following system of linear equations correct to two places by Gauss-Seidel method:
\[x + 4y + z = -1, \ 3x - y + z = 6, \ x + y + 2z = 4.\]

(b) Solve the differential equation \(u^2_x - u^2_y\) by variable separation method.

(c) In a steady fluid flow, the velocity components are \(u = 2kx, \ v = 2ky\) and \(w = -4kz\). Find the equation of a streamline passing through \((1, 0, 1)\).

Q7. (a) Solve the heat equation
\[\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < 1, \ t > 0\]
subject to the conditions \(u(0, t) = u(1, t) = 0\) for \(t > 0\) and \(u(x, 0) = \sin \pi x, \ 0 < x < 1\).

(b) Find the moment of inertia of a uniform mass \(M\) of a square shape with each side a about its one of the diagonals.

(c) Use the classical fourth order Runge-Kutta methods to find solutions at \(x = 0.1\) and \(x = 0.2\) of the differential equation \(\frac{dy}{dx} = x + y, \ y(0) = 1\) with step size \(h = 0.1\).

Q8. (a) Write a BASIC program to compute the product of two matrices.

(b) Suppose \(\vec{v} = (x - 4y)^i + (4x - y)^j\) represents a velocity field of an incompressible and irrotational flow. Find the stream function of the flow.

(c) Solve the wave equation \(\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\) for a string of length \(l\) fixed at both ends. The string is given initially a triangular deflection
\[u(x, 0) = \begin{cases} \frac{2}{l} x, & \text{if } 0 < x < \frac{l}{2} \\ \frac{2}{l} (l - x), & \text{if } \frac{l}{2} \leq x < l \end{cases}\]
with initial velocity \(u_t(x, 0) = 0\).