MATHEMATICS (PAPER-II)

Time Allowed : Three Hours

Maximum Marks : 250

QUESTION PAPER SPECIFIC INSTRUCTIONS

(Please read each of the following instructions carefully before attempting questions)

There are EIGHT questions divided in two Sections and printed both in HINDI and in ENGLISH.

Candidate has to attempt FIVE questions in all.

Question Nos. 1 and 5 are compulsory and out of the remaining, THREE are to be attempted choosing at least ONE question from each Section.

The number of marks carried by a question/part is indicated against it.

Answers must be written in the medium authorized in the Admission Certificate which must be stated clearly on the cover of this Question-cum-Answer (QCA) Booklet in the space provided. No marks will be given for answers written in medium other than the authorized one.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless otherwise indicated, symbols and notations carry their usual standard meanings.

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.
1. (a) (i) केरी 8 के चक्रीय यम पुण्ग ग के निलने जनक होते है? व्याख्या कीजिए।
How many generators are there of the cyclic group G of order 8? Explain.

(ii) केरी 4 के एक समूह {e, a, b, c} को लेते हुए, जहाँ e तत्समक (आइडेंटिटी) है, संग्रहेण सारणियाँ बनाइए 
यह दर्शाते हुए कि एक चक्रीय है जबकि दूसरी नहीं है।
Taking a group {e, a, b, c} of order 4, where e is the identity, construct 
composition tables showing that one is cyclic while the other is not.

(b) वलय का एक ऐसा उदाहरण दीजिए, जिसका तत्समक है परन्तु जिसके एक उपबलय का भिन्न तत्समक है।
Give an example of a ring having identity but a subring of this having a 
different identity.

(c) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1} \] के अभीसरण तथा निरस्त्र अभिसरण का परीक्षण कीजिए।
Test the convergence and absolute convergence of the series \[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2 + 1}. \]

(d) दर्शाए कि फलन \[ u(x, y) = \ln(x^2 + y^2) + x + y \] प्रसंबाद है। इसका संबंधी प्रसंबाद फलन \[ f(z) = u + iv \] को भी z के पदों में सात कीजिए।
Show that the function \[ u(x, y) = \ln(x^2 + y^2) + x + y \] is harmonic. Find its 
conjugate harmonic function \[ u(x, y). \] Also, find the corresponding analytic 
function \[ f(z) = u + iv \] in terms of \[ z. \]

(e) निम्नलिखित नियतन समस्या को अभीकल बिक्री करने के लिए हल कीजिए:
Solve the following assignment problem to maximize the sales:

<table>
<thead>
<tr>
<th>Territories (श्रेण)</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
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<tr>
<td>A</td>
<td>3</td>
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<td>B</td>
<td>4</td>
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<td>13</td>
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<tr>
<th>Salesmen (विक्रेता)</th>
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<tr>
<td>C</td>
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<td>D</td>
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<td>E</td>
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</table>
2. (a) If $R$ is a ring with unit element 1 and $\phi$ is a homomorphism of $R$ onto $R'$, prove that $\phi(1)$ is the unit element of $R'$.

(b) The function
\[ f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \leq \frac{1}{n} \\ 0, & x = 0 \end{cases} \]

Is the function $f(x)$ Riemann integrable? If yes, obtain the value of $\int_0^1 f(x) \, dx$.

(c) The function $f(z) = \frac{2z-3}{z^2-3z+2}$ is not defined at $z = 0$. However, it is defined in the vicinity of $z = 0$. Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \frac{2z-3}{z^2-3z+2}$ about the point $z = 0$.

3. (a) The Cauchy principal value of the integral
\[ \int_C \frac{e^z + 1}{z(z+1)(z-i)^2} \, dz; \quad C: |z| = 2 \]

Evaluate the integral using Cauchy's residue theorem.

(b) The series
\[ \sum_{n=1}^{\infty} \frac{nx}{(1 + n^2 x^2)} \]

Test the series for uniform convergence.
Consider the following linear programming problem:

Maximize \( Z = x_1 + 2x_2 - 3x_3 + 4x_4 \)

subject to

\[
\begin{align*}
    x_1 + x_2 + 2x_3 + 3x_4 &= 12 \\
    x_2 + 2x_3 + x_4 &= 8 \\
    x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

(i) Using the definition, find its all basic solutions. Which of these are degenerate basic feasible solutions and which are non-degenerate basic feasible solutions?

Without solving the problem, show that it has an optimal solution. Which of the basic feasible solution(s) is/are optimal?

4. (a) Do the following sets form integral domains with respect to ordinary addition and multiplication? If so, state if they are fields:

(i) \( \mathbb{Z} \sqrt{2} \) of the form \( b \sqrt{2} \) with \( b \) rational

The set of numbers of the form \( b \sqrt{2} \) with \( b \) rational

(ii) The set of even integers

(iii) The set of positive integers

(b) The set of numbers of the form \( b \sqrt{2} \) with \( b \) rational.
(c) निर्दिष्ट रैखिक प्रोग्राम समस्या को एक दिशा द्वारा हल कीजिए। इसकी दूसरी समस्या लिखिए। दी गई समस्या की इस्तमाल सारणी से दूसरी समस्या का इस्तमाल हल भी लिखिए:

अधिकतमीकरण कीजिए $Z = 2x_1 - 4x_2 + 5x_3$

बशर्ते कि

\[
\begin{align*}
    x_1 + 4x_2 - 2x_3 & \leq 2 \\
    -x_1 + 2x_2 + 3x_3 & \leq 1 \\
    x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Solve the following linear programming problem by the simplex method. Write its dual. Also, write the optimal solution of the dual from the optimal table of the given problem:

Maximize $Z = 2x_1 - 4x_2 + 5x_3$

subject to

\[
\begin{align*}
    x_1 + 4x_2 - 2x_3 & \leq 2 \\
    -x_1 + 2x_2 + 3x_3 & \leq 1 \\
    x_1, x_2, x_3 & \geq 0
\end{align*}
\]

खण्ड—B / SECTION—B

5. (a) आशिक अवकल समीकरण

\[
(y^2 + z^2 - x^2) p - 2xyq + 2xz = 0
\]

जहाँ $p = \frac{\partial z}{\partial x}$ तथा $q = \frac{\partial z}{\partial y}$, को हल कीजिए।

Solve the partial differential equation

\[
(y^2 + z^2 - x^2) p - 2xyq + 2xz = 0
\]

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

(b) $(D^2 + DD' - 2D'^2)u = e^{x+y}$ को हल कीजिए, जहाँ $D = \frac{\partial}{\partial x}$ तथा $D' = \frac{\partial}{\partial y}$.

Solve $(D^2 + DD' - 2D'^2)u = e^{x+y}$, where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$.

(c) बूलीय संलक $(p \land q \rightarrow r) \lor ((p \land q) \rightarrow -r)$ के लिए तीन चरों $p, q, r$ में मुख्य (अथवा विलिक्त) वियोजनीय (विकसितकरण) प्रसाधन रूप सत कीजिए। क्या दिया गया बूलीय संलक एक सिद्ध है या कि एक पुनरावृत्ति है?

Find the principal (or canonical) disjunctive normal form in three variables $p, q, r$ for the Boolean expression $((p \land q) \rightarrow r) \lor ((p \land q) \rightarrow -r)$. Is the given Boolean expression a contradiction or a tautology?
Consider a uniform flow $U_0$ in the positive $x$-direction. A cylinder of radius $a$ is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points.

Calculate the moment of inertia of a solid uniform hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$ with mass $m$ about the $OZ$-axis.

Solve for the general solution $p \cos(x + y) + q \sin(x + y) = z$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

Solve the plane pendulum problem using the Hamiltonian approach and show that $H$ is a constant of motion.

Find the Lagrange interpolating polynomial that fits the following data:

$x$ : -1 2 3 4
$f(x)$ : -1 11 31 69

Find $f(1.5)$.

Find $u(x,0) = x(l-x)$, $0 < x < l$.
Find the solution of the initial-boundary value problem

\[ u_t - u_{xx} + u = 0, \quad 0 < x < l, \quad t > 0 \]
\[ u(0, t) = u(l, t) = 0, \quad t \geq 0 \]
\[ u(x, 0) = x(l - x), \quad 0 < x < l \]

(b) Find a Lagrangian corresponding to this Hamiltonian.

(c) \[ \frac{dy}{dx} = x(y-x), \quad y(2)=3 \text{ in the interval } [2, 2.4] \]

Solve the initial value problem \[ \frac{dy}{dx} = x(y-x), \quad y(2)=3 \text{ in the interval } [2, 2.4] \]
using the Runge-Kutta fourth-order method with step size \( h = 0.2 \).

8. (a) Reduce the second-order partial differential equation

\[ x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \]

into canonical form. Hence, find its general solution.
(b) Find the solution of the system
\[
\begin{align*}
10x_1 - 2x_2 - x_3 - x_4 &= 3 \\
-2x_1 + 10x_2 - x_3 - x_4 &= 15 \\
x_1 - x_2 + 10x_3 - 2x_4 &= 27 \\
x_1 - x_2 - 2x_3 + 10x_4 &= -9
\end{align*}
\]
using Gauss-Seidel method (make four iterations).

(c) In an axisymmetric motion, show that stream function exists due to equation of continuity. Express the velocity components in terms of the stream function. Find the equation satisfied by the stream function if the flow is irrotational.